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COMMENT

Lacunarity and universality

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Abstract. A new expression for the lacunarity of Sierpinski carpets is proposed. It turns out that the universality can be described with the new expression, the fractal dimension and the connectivity.

Mandelbrot (1982) proposed the idea of fractals in order to describe the geometry of nature. A fractal differs from ordinary systems in having no characteristic length. Hence, physical systems on fractal lattices can behave in different manners from those on ordinary lattices. Recently, Gefen *et al* (1983a, b, 1984a, b) have investigated resistor networks and Potts models on several fractal lattices. In these studies they have found that only infinite ramified fractals (Gefen *et al* 1984a) have non-zero critical temperatures. Their infinite ramified fractals are Sierpinski carpets. For the carpets they have found that the critical exponents depend on the fractal dimension d_f, the connectivity Q and lacunarity L. Their results suggested the existence of universality classes. Hao and Yang (1987), however, have reported that the above three parameters—d_f, Q, L—are not sufficient to characterise the universality class, although Lin and Yang (1986) have improved the original lacunarity. In the present comment, we show that the above three parameters can classify the carpets according to universality in contrast to Hao and Yang (1987). For this purpose a new expression of lacunarity will be proposed instead of that proposed by Lin and Yang (1986).

We propose lacunarity which satisfies three conditions.

(i) Lacunarity $L=0$ if and only if the fractal is translationally invariant.

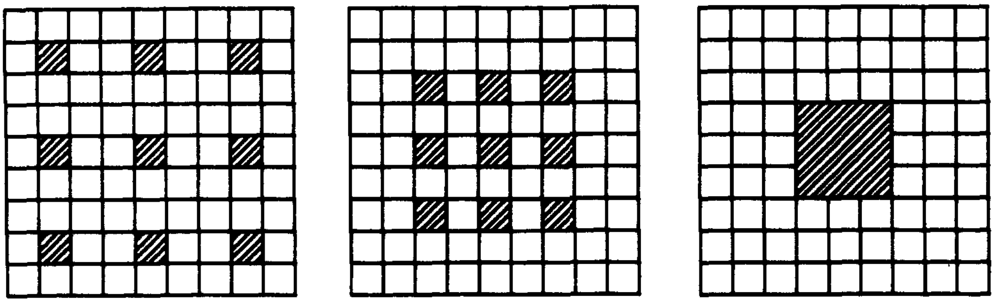
(ii) Lacunarity L decreases with increasing homogeneity of fractal.

(iii) If two systems have equal lacunarity, they belong to the same universality class.

The first condition is important, because Gefen *et al* (1983b) claimed that the translationally invariant Sierpinski carpets are hypercubics of the fractal dimension d_f. The next condition is not as important as the first one. However, this second condition is necessary for agreement with the original definition; lacunarity represents a degree of homogeneity (Gefen *et al* 1984a). The satisfaction of the last condition is the main purpose of this comment. Hence we consider condition (iii) first.

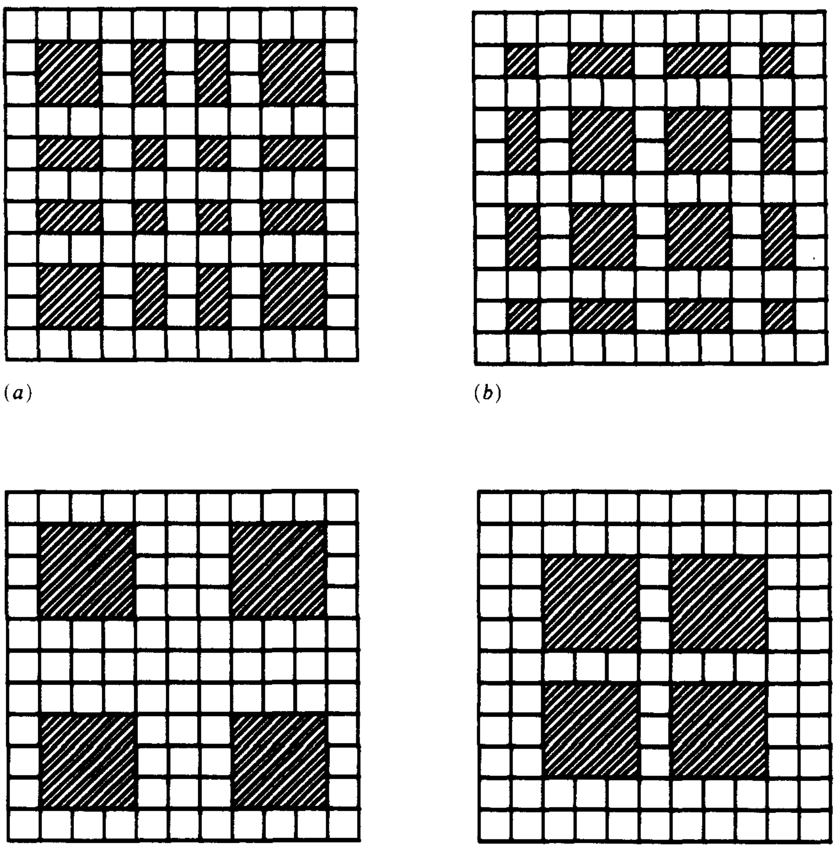
Condition (*iii*). The Sierpinski carpets are constructed as below. First we divide a square into $b\times b$ subsquares. Next, $l\times l$ subsquares are cut out in arbitrary manner: condensed or scattered (see figures 1 and 2). By introducing lacunarity L, we can measure the homogeneity of these carpets. Lin and Yang (1986) defined L as follows:

$$L=\frac{1}{N}\sum_{s=l}^{b} L(s) \qquad (1)$$



(a) (b) (c)

Figure 1. Three types of the first stages of the Sierpinski carpet with $b=9$, $l=3$, fractal dimension $d_f=1.946$ and connectivity $Q=0.815$. (a) and (b) belong to the same universality class while (c) belongs to a different one.



(a) (b) (c) (d)

Figure 2. Four types of the first stages of the Sierpinski carpet with $b=11$, $l=6$, fractal dimension $d_f=1.853$ and connectivity $Q=0.671$. (a) and (b) belong to the same universality class, and (c) and (d) belong to a different one.

where $N = b - l + 1$. $L(s)$ is related to $L'(s)$,

$$L(s) = \frac{1}{\bar{n}} (L'(s))^{1/2} \tag{2}$$

where $L'(s) = (1/n)\Sigma_i (n_i - \bar{n})^2$. n is the number of square subarrays of $s \times s$ cells in an array of $b \times b$ cells. It can easily be seen that $n = (b - s + 1)^2$ and n_i represents the number of non-eliminated subsquares in the ith $s \times s$ covering. $\bar{n}$ is the average number of n_i ($\bar{n} = \Sigma_i n_i / n$) and the original lacunarity (Gefen *et al* 1984a) is equal to $L'(l)$. Figures 1(a)-(c) show some examples of Sierpinski carpets.

According to Hao and Yang (1987), figures 1(a) and (b) belong to the same universality class in accordance with the approximation of the bond-moving renormalisation. However, as shown in table 1, the carpet in figure 1(a) has different values of lacunarity L from that in figure 1(b). Here we introduce a new lacunarity,

$$L^{(m)} = \frac{1}{(b-1)} \sum_{s=2}^{b} L^{(m)}(s). \tag{3}$$

The average of $L^{(m)}(s)$ is taken over an array of $b^m \times b^m$ cells, i.e. m stage of the construction, instead of $b \times b$ cells. Table 2 shows values of lacunarity for $m = 2$. It is easily seen that new lacunarity can describe universality.

Table 1. Values of L for carpets in figures 1(a)-(c). According to Hao and Yang (1987), the carpets in figures 1(a) and (b) belong to the same universality class. However, the values of lacunarity for these two carpets are different. Thus L fails to describe a universality class.

	Figure		
	1(a)	1(b)	1(c)
s	$L(s)$	$L(s)$	$L(s)$
3	0	0.1152	0.2930
4	0.0644	0.0857	0.2003
5	0.0474	0.0622	0.0996
6	0	0.0600	0
7	0.0361	0	0
8	0	0	0
9	0	0	0
L	0.0211	0.0462	0.0847

Condition (ii). According to Lin and Yang (1986), their lacunarity represents the homogeneity better than the original one (Gefen *et al* 1984a). However, there are some special exceptions. Figures 2(a)-(d) show such exceptions. Intuitively, figure 2(a) is more homogeneous than figure 2(c). Nevertheless, according to Lin and Yang (1986), figure 2(c) is more homogeneous than figure 2(a), because the value of lacunarity of figure 2(c) is smaller than that of figure 2(a) (see table 3). In contrast to this, our new expression of lacunarity agrees with the intuitive idea of homogeneity (see table 4). We conclude, therefore, that our lacunarity, $L^{(2)}$, is more suitable to measuring the homogeneity than that of Lin and Yang (1986). In addition to this, our lacunarity also describes the universality for figures 2(a)-(d). The bond-moving renormalisation

Table 2. Values of $L^{(2)}$ for the same carpets as in table 1. $L^{(2)}$ can describe a universality class.

	Figure		
	1(*a*)	1(*b*)	1(*c*)
s	$L^{(2)}(s)$	$L^{(2)}(s)$	$L^{(2)}(s)$
2	0.362	0.362	0.466
3	0.320	0.340	0.429
4	0.315	0.314	0.396
5	0.300	0.296	0.374
6	0.283	0.280	0.359
7	0.272	0.263	0.350
8	0.256	0.249	0.345
9	0.239	0.233	0.344
$L^{(2)}$	0.293	0.292	0.383

Table 3. Values of L for the carpets in figures 2(*a*)-(*d*). A glance at these figures tells us that figure 2(*a*) is more homogeneous than figure 2(*c*). However, the values of L lead to an opposite conclusion: figure 2(*c*) is more homogeneous than figure 2(*a*).

	Figure			
	2(*a*)	2(*b*)	2(*c*)	2(*d*)
s	$L(s)$	$L(s)$	$L(s)$	$L(s)$
6	0.0897	0.1088	0	0.2333
7	0	0.1032	0.0695	0.1969
8	0.0730	0	0.0730	0.1155
9	0.0678	0.0678	0.0678	0
10	0	0	0	0
11	0	0	0	0
L	0.0384	0.0466	0.0350	0.0910

(Hao and Yang 1987) shows that figures 2(*a*) and (*b*) should belong to one universality class and figures 2(*c*) and (*d*) to another. As shown in table 4, our lacunarities take almost equal values for carpets belonging to the same universality class.

Condition (*i*). In contrast to the original lacunarity (Gefen *et al* 1984a), equation (3) cannot assume a zero when the carpets are inhomogeneous. The reason quoted by Lin and Yang (1986) can also be applied to our new expression (3).

In conclusion, our new expression (3) can satisfy the above three conditions. Next we see why our new expression can satisfy the above three conditions better than that of Lin and Yang (1986).

The Sierpinski carpet is one of the fractal lattices which obey the scaling concept. Hence, we should discuss lacunarity with scaling theory. Here we introduce a lacunarity scaling function $\mathscr{L}^{(m)}(x)$ as follows:

$$\mathscr{L}^{(m)}(x) \equiv L^{(m)}(s) \qquad x = sb^{-m}. \tag{4}$$

Table 4. Values of $L^{(2)}$ for the same carpets as in table 3. The homogeneity given by $L^{(2)}$ agrees with the intuitive impression. Moreover, the almost constant value of $L^{(2)}$ implies that the carpets belong to the same universality class.

	Figure			
	2(*a*)	2(*b*)	2(*c*)	2(*d*)
s	$L^{(2)}(s)$	$L^{(2)}(s)$	$L^{(2)}(s)$	$L^{(2)}(s)$
2	0.695	0.695	0.864	0.864
3	0.626	0.626	0.757	0.773
4	0.597	0.612	0.677	0.703
5	0.578	0.571	0.643	0.670
6	0.564	0.551	0.638	0.650
7	0.542	0.534	0.642	0.634
8	0.526	0.507	0.639	0.621
9	0.504	0.490	0.628	0.608
10	0.481	0.468	0.616	0.600
11	0.456	0.445	0.609	0.595
$L^{(2)}$	0.557	0.550	0.671	0.672

Figure 3 shows the dependence of $\mathscr{L}^{(m)}(x)$ upon m for $b=3$, $l=1$. It suggests that

$$\lim_{m\to\infty} \frac{\mathscr{L}^{(m+1)}(x)}{\mathscr{L}^{(m)}(x)} = 1. \qquad (5)$$

The main reason for the inappropriateness of the argument of Lin and Yang (1986) is that they set $m=1$. m must take a large enough value to reach the asymptotic region of $\mathscr{L}^{(m)}(x)$. Setting $m=2$ in the range $0<x\leqslant b^{-1}$ we can get enough accurate values of $\mathscr{L}^{(m)}(x)$; they provide lacunarity $L^{(m)}$ satisfying the above conditions. If we desire a greater accuracy than equation (3), we should make m larger than 2.

In conclusion, we have proposed a new expression of lacunarity $L^{(m)}$ which satisfies the following conditions.

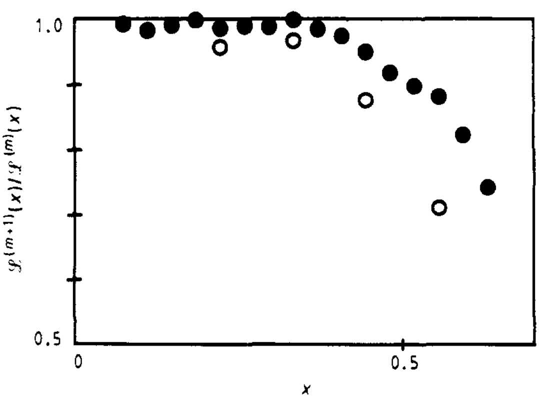


Figure 3. Ratios of $\mathscr{L}^{(m+1)}(x)$ to $\mathscr{L}^{(m)}(x)$ for $b=3$, $l=1$ and $m=2$ (○) and 3 (●). It is easily seen that $\mathscr{L}^{(m)}(x)$ can converge with an increasing value of m. Therefore m must take a large enough value to reach the asymptotic region of $\mathscr{L}^{(m)}(x)$.

(i) $L^{(m)} = 0$ if and only if the fractal is translationally invariant.

(ii) $L^{(m)}$ decreases with increasing homogeneity of fractal.

(iii) If two systems have the same value of lacunarity, they belong to the same universality class.

Due to condition (iii), using our new expression of lacunarity with d_f and Q, we can describe a universality class of Potts models on the carpets.

References

Gefen Y, Aharony A and Mandelbrot B B 1983a *J. Phys. A: Math. Gen.* **16** 1267
Gefen Y, Meir Y, Mandelbrot B B and Aharony A 1983b *Phys. Rev. Lett.* **50** 145
Gefen Y, Aharony A and Mandelbrot B B 1984a *J. Phys. A: Math. Gen.* **17** 1277
Gefen Y, Aharony A, Shapir Y and Mandelbrot B B 1984b *J. Phys. A: Math. Gen.* **17** 435
Hao L and Yang Z R 1987 *J. Phys. A: Math. Gen.* **20** 1627
Lin B and Yang Z R 1986 *J. Phys. A: Math. Gen.* **19** L49
Mandelbrot B B 1982 *The Fractal Geometry of Nature* (San Francisco: Freeman)